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V.N.Abrashin

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CONVERGENCE AND SQUARE MEAN ERROR ESTIMATION IN SOLUTION OF PARTICULAR BOUNDARY-VALUE PROBLEMS FOR ELLIPTICS EQUATIONS BY THE STRAIGHT-LINE METHOD

V.N.Abrashin

ABSTRACT. The convergence and square mean error estimate are given. The pattern of the method was considered with two independent variables. However, this can also be generalized with more than two, and here the method becomes a hyperplane one.

In the article there is proven the convergence, and there is given a mean-square error estimate, in the solution of some boundary problems for elliptical equations by the straight-line method. An indication is given for the possible generalization of the obtained results to the hyperplane method (a multidimensional analogue of the straight-line method). /670 *

Let there be required to solve a differential equation of the elliptical type

$$L(u) = (p(x)u_x)_x - d(x)u = f(x, y) - u_{yy} \tag{1}$$

with the following boundary conditions in contour Γ of the rectangular integration area Π $(0 < x < a,\ 0 < y < b)$:

$$u|_{y=0} = \varphi_1(x),\quad u|_{y=b} = \varphi_2(x), \tag{2}$$

$$D_1(u) = \alpha_1 u + \beta_1 u_x|_{x=0} = \psi_1(y), \tag{3}$$

$$D_2(u) = \alpha_2 u + \beta_2 u_x|_{x=a} = \psi_2(y),$$

$$\alpha_1\beta_1 \leqslant 0,\quad \alpha_2\beta_2 \geqslant 0,\quad |\alpha_1| + |\beta_1| \neq 0,\quad |\alpha_2| + |\beta_2| \neq 0,$$

where $p,\ p',\ d \in C[0, a],\ p > 0,\ d \geqslant 0$.

Using the interpolation formula cited in the article by Sh. Ye. Mikeladze (Ref. 1), problem (1) - (3) shall be reduced to a solution of a boundary problem for a system of conventional differential equations

$$L(u_{n+1}) + 10L(u_n) + L(u_{n-1}) = -h^{-2}12(u_{n+1} - 2u_n + u_{n-1}) + f_{n+1} + 10f_n + f_{n-1},$$

$$0 < x < a,\quad u_0 = \varphi_1(x),\quad u_N = \varphi_2(x),\quad D_i(u_n) = \psi_i(y_n),\quad i = 1, 2,$$

$$n = 1, 2, \ldots, N-1,\quad f_n = f(x, y_n)$$

with respect to functions $u_n(x)$ $(0 < x < a)$ on straight lines $y = y_n = nh$, $n = 0, 1, \ldots, N$; $y_0 = 0$, $y_N = b$.

In view of the linearity of operators L, D_1 and D_2, the errors of the method $\gamma_n(x) = u(x, y_n) - u_n(x)$, appear in the solutions of the problem:

$$L(\gamma_{n+1}) + 10L(\gamma_n) + L(\gamma_{n-1}) = -h^{-2}12(\gamma_{n+1} - 2\gamma_n + \gamma_{n-1}) + 12r_n,$$

$$0 < x < a,\quad \gamma_0 = \gamma_N = 0,\quad D_1(\gamma_n) = D_2(\gamma_n) = 0,\quad n = 1, 2, \ldots, N-1, \tag{4}$$

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* Numbers in the margin indicate pagination in the foreign text.

where

$$r_n = 2h^4 \int_0^1 (1-t)\, t^2 (t^2-1)\, u_{yy}(x,\ y_n \pm th,\ y_n,\ y_n \pm h)\, dt,$$

the symbol $u_{yy}(x,\ y_n \pm th,\ y_n,\ y_n \pm h)$ designates the fourth divided difference of function $u_{yy}(x,\ y_n + th)$.

It is known (Ref. 2) that the spectrum of the operator L(y) with the conditions $D_1(y) = D_2(y) = 0$ is real and discrete. The eigenfunctions may be considered to be orthonormali:

$$L(\varphi_k) = -\lambda_k \varphi_k, \quad k = 1,\ 2,\ \ldots,\ (\varphi_k,\ \varphi_l) = \delta_{kl}.$$

Let us multiply equation (4) by the Green function $G(x,\ \xi)$ which corresponds to the operator, and integrate from 0 to a:

$$\int_0^a G(x,\ \xi)\left[L(\gamma_{n+1}(\xi) + 10L(\gamma_n) + L(\gamma_{n-1})\right] d\xi =$$

$$= \int_0^a G\left[-h^{-2}12(\gamma_{n+1} - 2\gamma_n + \gamma_{n-1}) + 12r_n\right] d\xi,$$

$$n = 1,\ 2,\ \ldots,\ N-1.$$

If $r_n,\ \gamma_0,\ \gamma_0'',\ \gamma_N,\ \gamma_N'' \in L_2[0,\ a]$, then $\gamma_n(x)$ $(n = 0,\ 1,\ \ldots,\ N)$ are sourcewise representable in terms of $G(x,\ \xi)$ and consequently, can be expanded by [0, a] in a regularly converging series according to the eigenfunctions.

For the Fourier coefficients

$$\alpha_{nk} = (\gamma_n,\ \varphi_k) = \int_0^a \gamma_n(x)\, \varphi_k(x)\, dx,$$

$$\beta_{nk} = (r_n,\ \varphi_k), \quad n = 1,\ 2,\ \ldots,\ N,\ k = 1,\ 2,\ \ldots,$$

we obtained the differential equation

$$\left(1 - h^2\lambda_k \frac{1}{12}\right)\alpha_{n+1k} - \left(2 + h^2\lambda_k \frac{10}{12}\right)\alpha_{nk} + \left(1 - h^2\lambda_k \frac{1}{12}\right)\alpha_{n-1k} = h^2\beta_{nk}, \tag{5}$$

$$n = 1,\ 2,\ \ldots,\ N-1,\ k = 1,\ 2,\ \ldots$$

With the restrictions imposed upon $d,\ \alpha_i,\ \beta_i$ $(i = 1,\ 2)$, the values of λ_k will be non-negative and, consequently, for (5) the inequality

$$2|\alpha_{nk}| \leqslant |\alpha_{n+1k}| + |\alpha_{n-1k}| + h^2|\beta_{nk}|. \tag{6}$$

is valid whence, we successively find

$$|\alpha_{1k}| \leqslant |\alpha_{2k}| + h^2 |\beta_{1k}|,$$

$$|\alpha_{2k}| \leqslant |\alpha_{3k}| + h^2 \sum_{m=1}^{2} |\beta_{mk}|,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$|\alpha_{jk}| \leqslant |\alpha_{j+1k}| + h^2 \sum_{m=1}^{j} |\beta_{mk}|, \tag{7}$$

$$j = 1, 2, \ldots, N-1.$$

Summating (7), starting with j = N - 1, we obtain /672

$$|\alpha_{N-jk}| \leqslant h^2 \sum_{i=1}^{j} \sum_{m=1}^{N-i} |\beta_{mk}|, \quad j = 1, 2, \ldots, N-1. \tag{8}$$

According to the inequality of Cauchy-Bunyakovskiy, from (8), we have

$$\alpha_{N-jk}^2 \leqslant b (jh)^2 h \sum_{m=1}^{N-1} \beta_{mk}^2, \quad j = 1, 2, \ldots, N-1.$$

We summate the last inequality for all values of k, and since

$$\sum_k \alpha_{nk}^2 = \|\gamma_n\|^2 = (\gamma_n, \gamma_n), \quad \sum_k \beta_{nk}^2 \leqslant \|r_n\|^2,$$

$$n = 0, 1, \ldots, N, \quad k = 1, 2, \ldots,$$

we obtain

$$\|\gamma_{N-j}\|^2 \leqslant (jh)^2 bh \sum_{m=1}^{N-1} \|r_m\|^2, \quad j = 1, 2, \ldots, N-1. \tag{9}$$

Substituting the values of the remainder into (9), we conclude that when $h \to 0$ at each straight line $y = y_n (n = 1, 2, \ldots, N-1)$, the order of error will be as follows:

$$\|\gamma_n\|^2 \leqslant (nh)^2 ((N-n)h)^2 O(h^5).$$

If we consider systems of the form

$$L(u_n) = f_n - h^{-2} (u_{n+1} - 2u_n + u_{n-1}),$$

for which uniform convergence has been proved, it follows that the mean-square error will have the following order:

$$\|\gamma_n\|^2 \sim (nh)^2 ((N-n)h)^2 O(h^5), \quad n = 1, 2, \ldots, N-1.$$

We have considered the pattern of the straight-line method for an elliptical equation with two independent variables. The obtained result can also be generalized for the case of an equation with a number of independent variables greater than two. Here the straight-line method passes into the so-called hyper-plane method.

Institute of Mathematics, Academy of Sciences
Belorussian SSR
March 14, 1967

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